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An elementary mean-field approach to the spectral densities of random matrix ensembles

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ABSTRACT

We present a simple mean-field approach for calculating spectral densities for random matrix ensembles in the thermodynamic limit. Our approach is based on constructing a linear system of equations and calculating how the solutions to these equation change in response to a small perturbation using the zero-temperature cavity method. We illustrate the power of the method by providing simple analytic derivations of the Wigner Semi-circle Law for symmetric matrices, the Marchenko–Pastur Law for Wishart matrices, the spectral density for a product Wishart matrix composed of two square matrices, and the Circle and elliptic laws for real random matrices.

Random matrices are central to variety of problems ranging from statistical physics [1,2] to quantum chaos [3], ecology [4–6], and wireless communication [7]. An important problem in Random Matrix Theory (RMT) is to calculate the spectral density of an ensemble of random matrices [8]. The spectral density measures the density of eigenvalues in the complex plane and plays a central role in many RMT-based approaches. There exist numerous methods for calculating spectral densities, including many prominent methods that draw heavily from the physics of disordered systems such as the replica method [9], large-N diagrammatic expansions [10], and the finite temperature cavity method [11,12].

Here, we introduce a simple mean-field approach for calculating spectral densities in the thermodynamic limit (i.e., in the limit where the size of the matrices become infinitely large). Our method is inspired by recent zero temperature cavity calculations in the context of ecology [13–16]. The central observation underlying the method is that the spectral density can be calculated by constructing an appropriately chosen system of random linear equations and then asking how the solutions to these equations change in response to small constant perturbations. In particular, we show that the trace of the susceptibility matrix which measures responses to perturbations is precisely the resolvent or Green's function and can be calculated easily using the zero-temperature cavity method. Since at its core of our approach is just simply a mean-field way to evaluate Green's functions, we can make use of many powerful results in RMT and statistical physics relating Green's functions to spectral densities. These include generalizations of the Green's function method to non-Hermitian matrices through hermitian reduction [17,18], which associates with each N-dimensional non-hermitian ensemble an auxiliary ensemble of $2N$-dimensional Hermitian matrices. In our approach, this "doubling" of the degrees of freedom simply corresponds to "doubling" the number of random equations, allowing us to also easily calculate spectral densities of many non-Hermitian ensembles.

Our approach is closely related to the Stieltjes transform method and the finite temperature cavity method. The former is a standard approach in RMT which employs the Schur complement to directly obtain the resolvent by deleting one row and one

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column from the original random matrix [19–21]. The latter formalizes the problem in terms of the Gibbs–Boltzmann probability distribution and uses the finite temperature cavity method to estimate the marginal probability [11,12]. Our approach is analogous to deleting one row and one column from a matrix in the Stieltjes transform method, while the set of linear equations we construct can be interpreted as a result of taking the "zero-temperature" limit in the finite temperature cavity method. By combining aspects of these two approaches to develop a method with the same range of applicability, but that we often find in practice to be simpler to perform and easier to establish the connection between RMT spectrum to susceptibilities to external perturbations, which turns out very useful in the linear stability analysis of dynamical systems [16].

The connection between linear systems and the resolvent is also explored in [22,23], with a focus on addressing low-rank perturbations. In such cases, the bulk spectrum remains unchanged, while outlier eigenvalues may emerge. However, it is important to note that our method is not applicable when the resolvent matrices are non-diagonal, as discussed in [22,24]. Baron et al. developed diagrammatic techniques within the path integral formalism, particularly for random matrices with novel correlations, and overcome the limitation of our approach.

Our approach assume a Replica Symmetric Ansatz and exploits the zero-temperature cavity method to calculate Green's functions [25]. An important technical consideration that makes the method particularly simple to implement is that, for many random matrix ensembles, there is no need to explicitly solve the resulting self-consistent mean-field cavity equations. Instead, the problem often reduces to simply solving a polynomial equation for the susceptibility. This advantage is notable in the case of products of random matrices, whose mean-field equations can be linearized easily by introducing additional variables. While replica or path-integral methods typically involves Hubbard–Stratonovich transformation along with the introduction of "order parameters" [22,26] . Our approach presented here offers a simpler alternative.

In this paper, we illustrate our approach by giving simple derivations of spectral densities for a number of random matrix ensembles. We begin by giving some background on the resolvent/Green's function method for Hermitian and non-Hermitian matrices. We then introduce our approach in the context of real symmetric random matrices and show how to derive the Wigner's semi-circle law [27]. We then show how this construction can be generalized to calculate the Marchenko–Pastur Law for Wishart matrices [28] and to calculate the spectral density for a product Wishart matrix composed of two square matrices. Finally, we show how our method can be generalized to real non-symmetric matrices and provide simple derivations of Girko's circle law [29] and elliptic laws [30] for real random matrices .

1. Resolvent methods for Hermitian and non-Hermitian matrices

We begin by briefly summarizing the mathematical results we make use of in the paper. For a full discussion, of the Resolvent/Green's function method in RMT we urge the reader to consult one of the many excellent reviews or textbooks in the field [7,8,21,31–33].

1.1. Hermitian matrices

Let A be a $N \times N$ symmetric matrix. The resolvent or Green's function of A is given by the expression

$$G_A(z) = \frac{1}{zI - A}. \tag{1}$$

where z is a scalar constant and I is the $N \times N$ identity matrix. We define the spectral density of A as

$$\rho_A(x) = \frac{1}{N} \sum_i \delta(x - \lambda_i), \tag{2}$$

where λ_i are the eigenvalues of the matrix A. It what follows we will be concerned almost exclusively with the thermodynamic limit where $N \to \infty$ and the spectrum becomes continuous. The spectral density ρ_A of the eigenvalues of A can be extracted from the Green's function using the standard relationship

$$\rho_A(x) = \lim_{\epsilon \to 0^+} \frac{1}{\pi} \mathrm{Im}\Big[\frac{1}{N} \mathrm{Tr}\, G_A(x - i\epsilon)\Big], \tag{3}$$

where Tr is the trace.

1.2. Non-Hermitian matrices

Feinberg and Zee [17] generalized this to non-Hermitian matrices (which we also denote by A) by carefully considering differentials in the complex plane. Generically, the eigenvalues λ_i of a non-hermitian matrix are complex and have non-zero real and imaginary parts. The spectral density of A over the complex plane $z = x + iy$ is given by

$$\rho_A(x, y) = \frac{1}{N} \sum_i \delta(x - \mathrm{Re}[\lambda_i])\delta(y - \mathrm{Im}[\lambda_i]). \tag{4}$$

Again, we will focus on the thermodynamic limit where $N \to \infty$.

Feinberg and Zee exploited the fact that over the complex plane on a single Riemann cut

$$\partial_z \frac{1}{z^*} = \pi\delta(x)\delta(y) \tag{5}$$

to generalize the resolvent formalism above to calculate such spectral functions. In particular, they considered a "symmetrization" of the matrix A by defining a new $2N \times 2N$ Hermitian matrix

$$H(z) = \begin{pmatrix} 0 & A - zI \\ A^* - z^* I & 0 \end{pmatrix} \tag{6}$$

and a corresponding Green's function in the $2N$ dimensional space

$$\mathcal{G}_A(\eta) = \frac{1}{\eta I - H} \tag{7}$$

where η is a scalar constant. This Green's function can be rewritten in block diagonal form (with each block matrix is of size $N \times N$) as

$$\mathcal{G}_A(\eta) = \begin{pmatrix} \chi & \nu \\ \nu^* & \chi \end{pmatrix}, \tag{8}$$

where in writing this in terms of χ and ν we have exploited the symmetries and block diagonal structure of $H(z)$. In terms of block matrices, these relationships can be written as

$$\begin{pmatrix} \eta & zI - A \\ z^* I - A^* & 0 \end{pmatrix} \begin{pmatrix} \chi & \nu \\ \nu^* & \chi \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}. \tag{9}$$

Focusing on the upper left block, one has $\eta\chi + (zI - A)\nu^* = I$, which for the special case $\eta = 0$ implies that the resolvent (Eq. (13)) is

$$G_A(z) = \frac{1}{zI - A} = \nu^* \tag{10}$$

and

$$\rho_A(x, y) = \frac{1}{\pi} \partial_{z^*} \left[\frac{1}{N} \mathrm{Tr}\ \nu^*(z, z^*) \right] \Big|_{\eta=0}, \tag{11}$$

where in going to the second equation we make use of Eq. (5).

Here, we slightly modify this prescription. The reason is that a careful analysis actually shows that this formalism, while essentially correct, results in choosing the wrong complement of the full Riemann sphere for the domain where the spectral density is non-zero (though the boundary between regions is correctly identified). We note that this mistake is already present in the explicit expressions for the Circle law derived in [17]. To correct this mistake, we modify a sign in the original construction and consider the $2N$-dimensional non-symmetric matrix

$$H(z) = \begin{pmatrix} 0 & A - zI \\ -A^* + z^* I & 0 \end{pmatrix} \tag{12}$$

and once again calculate the corresponding Green's function in the $2N$ dimensional space

$$\mathcal{G}_A(\eta) = \frac{1}{\eta I - H}. \tag{13}$$

This Green's function can be rewritten in block diagonal form (with each block matrix is of size $N \times N$) as

$$\mathcal{G}_A(\eta) = \begin{pmatrix} \chi\nu & \\ -\nu^* & \chi \end{pmatrix}, \tag{14}$$

and an almost identical calculation yields that

$$\rho_A(x, y) = -\frac{1}{\pi} \partial_{z^*} \left[\frac{1}{N} \mathrm{Tr}\ \nu^*(z, z^*) \right] \Big|_{\eta=0} \tag{15}$$

This sign change ensures that the domain where the spectral density is non-zero density occurs on the correct complement of the full Riemann sphere [12,34].

2. The Wigner semi-circle law

We start to derive the spectrum for the ensemble of real, symmetric random matrices A with our method. We assume that the entries of A, denoted A_{ij} $(i, j = 1, \ldots, N)$, are independent (up to symmetry) with mean and variance of the form

$$\langle A_{ij} \rangle = 0, \quad \langle A_{ij} A_{kl} \rangle = \frac{\sigma^2}{N} (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}). \tag{16}$$

Wigner showed that the spectral density of this ensemble is described by the semi-circle law shown in Fig. 1 [27]. More explicitly, in thermodynamic limit the spectral density is just

$$\rho_A(x) = \begin{cases} \frac{1}{2\pi\sigma^2} \sqrt{4\sigma^2 - x^2} & \text{if} \quad |x| \le 2\sigma \\ 0 & \text{if} \quad |x| > 2\sigma. \end{cases} \tag{17}$$

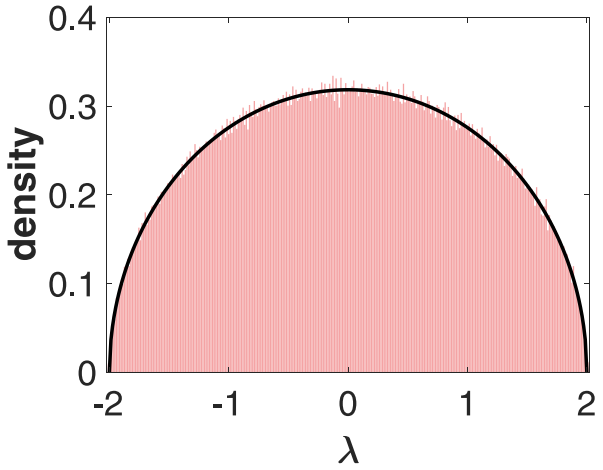

Fig. 1. Wigner semi-circle law for symmetric matrices. The spectrum of a $N \times N$ symmetric random matrix A whose entries are *i.i.d.* (up to symmetry) with mean zero and variance σ^2/N for $\sigma^2 = 1$ and $N = 1000$ with 100 independent realizations [see Eq. (16)]. The solid black line shows analytic predictions of the Wigner semi-circle law which is exact in the thermodynamic limit $N \to \infty$.

2.1. Resolvent equations

To calculate this spectral density, we first construct a system of equations that will allow us to solve for the Green's function $G_A(z)$ of Eq. (1). We propose the following system of N equations for N unknown variables u_i:

$$zu_i = \sum_{j=1}^{N} A_{ij}u_j + a_i. \tag{18}$$

We have also introduced the constant z along with N constant auxiliary variables a_i.

To see how this system of equation encodes $G_A(z)$, we ask how the solutions change if we add a small, independent perturbation δa_i to each of the a_i. In the our method, the response to these perturbations is characterized by the $N \times N$ "susceptibility matrix"

$$\nu_{ij} = \frac{\partial u_i}{\partial a_j}. \tag{19}$$

It is straightforward to show that this susceptibility matrix is equivalent to $G_A(z)$.

To see this, we take the derivative of Eq. (18) with respect to a_k to get

$$z\nu_{ik} = \sum_{j=1}^{N} A_{ij}\nu_{jk} + \delta_{ik} \tag{20}$$

where δ_{ik} is just the Kronecker-Delta function. In matrix form, this set of equations can be written as

$$(zI - A)\nu(z) = I \tag{21}$$

where I is the $N \times N$ identity matrix. Equivalently, we can write

$$\nu(z) = \frac{1}{zI - A} \tag{22}$$

which we see is identical to Eq. (1), showing that the Green's function $G_A(z)$ is exactly the susceptibility of the linear equations above. In particular, we are interested in the trace of the susceptibility

$$\bar{\nu} = \frac{1}{N}\sum_{j=1}^{N} \nu_{jj}(z) \tag{23}$$

which allows us to calculate the spectral density using Eq. (3).

2.2. Cavity expansion

To calculate the susceptibility, we assume replica symmetry and make use of the zero-temperature cavity method. In the cavity method, one relates a system of N equations of the N variables u_i to a system with $N+1$ equations and $N+1$ variables. By convention,

we denote the additional variable by u_0 and the additional rows and columns of the matrix A by A_{0i} and A_{0j}, respectively. In the presence of these new variables, Eq. (18) becomes

$$zu_i = \sum_j A_{ij}u_j + a_i + A_{i0}u_0 \tag{24}$$

(Note: indices are implied to range from 1 to N and terms with 0-valued indices always be specified explicitly).

From Eq. (16), we know that the extra terms $A_{i0}u_0$ scale as $1/N$. Therefore, we treat these extra terms as small perturbations to the original equations via the auxiliary variables a_i,

$$\delta a_i = A_{i0}u_0, \tag{25}$$

and treat them using perturbation theory. We also know that the solutions u_i to Eq. (24) must be related to the solutions $u_{i\setminus 0}$ of Eq. (18) (i.e., without the new variable u_0) by the perturbative relation

$$\begin{aligned} u_i &\approx u_{i\setminus 0} + \sum_j \nu_{ij}\delta a_j \\ &= u_{i\setminus 0} + u_0 \sum_j \nu_{ij}A_{j0} \end{aligned} \tag{26}$$

where we have made use of the definition of the susceptibility [Eq. (19)].

Now let us turn to additional equation corresponding to the new row in the matrix:

$$zu_0 = \sum_j A_{0j}u_j + a_0 + A_{00}u_0 \tag{27}$$

Substituting the perturbative expansion above, we get

$$zu_0 = \sum_j A_{0j}u_{j\setminus 0} + u_0 \sum_{jk} \nu_{jk}A_{0j}A_{k0} + a_0 + A_{00}u_0 \tag{28}$$

2.3. Approximation via central limit theorem

In the next step, we approximate the sum in Eq. (28) that includes the susceptibility matrix. In this sum, the elements of the susceptibility matrix are statistically independent of the new elements of A (those with at least one 0-valued index). In addition, this sum includes a very large (order N) number of statistically independent terms. Due to these two properties, it is straightforward to show using the central limit theorem that this sum will be dominated by its mean with respect to the new elements of A. Performing this average, we find that the sum takes the form

$$\begin{aligned} \sum_{jk} \nu_{jk}A_{0j}A_{k0} &\approx \sum_{jk} \nu_{jk} \left\langle A_{0j}A_{k0} \right\rangle \\ &= \frac{\sigma^2}{N} \sum_{jk} \nu_{jk}\delta_{jk} \\ &= \sigma^2 \bar{\nu} \end{aligned} \tag{29}$$

where $\bar{\nu}$ is the trace of the susceptibility defined in Eq. (23).

2.4. Self-consistency equation for susceptibility

Applying the approximation from the previous section we rewrite Eq. (28) as

$$zu_0 \approx \sum_j A_{0j}u_{j\setminus 0} + u_0\sigma^2\bar{\nu} + a_0 \tag{30}$$

where we have also dropped the term $A_{00}u_0$ because it is small (order $1/N$). Rearranging we find the following expression for u_0:

$$u_0 = \frac{\sum_j A_{0j}u_{j\setminus 0} + a_0}{1 - \sigma^2\bar{\nu}}. \tag{31}$$

Thus, u_0 is a Gaussian random variable. By definition, we know that

$$\langle \nu_{00} \rangle = \left\langle \frac{\partial u_0}{\partial a_0} \right\rangle = \frac{1}{1 - \sigma^2\bar{\nu}}. \tag{32}$$

However, since there is nothing special about u_0 (i.e., the system self-averages), it is evident that

$$\bar{\upsilon} = \frac{1}{N} \sum_j \nu_{jj} \approx \langle \nu_{00} \rangle . \tag{33}$$

This gives us a self-consistency equation for $\bar{\upsilon}$ of the form

$$\bar{\nu} = \frac{1}{z - \sigma^2\bar{\nu}} \tag{34}$$

or equivalently, the quadratic equation

$$\sigma^2\bar{\nu}^2 - z\bar{\nu} + 1 = 0. \tag{35}$$

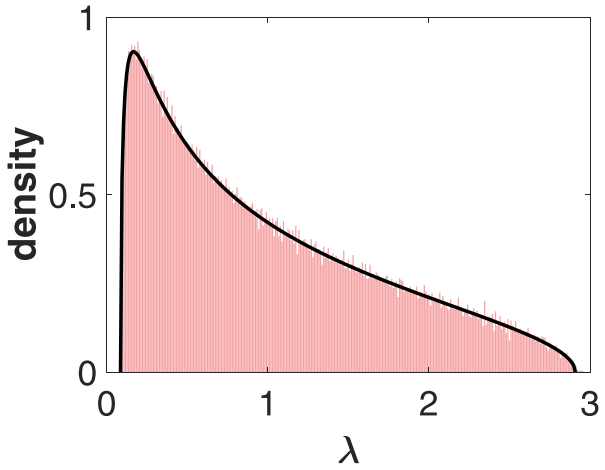

Fig. 2. Marchenko Pastur law for Wishart matrices. The spectrum of a Wishart random matrix of the form $A = CC^T$ where C is an $M \times N$ matrix with entries *i.i.d.* entries $C_{i\alpha}$ with mean zero and variance σ^2/N for $\sigma = 1$, $M = 500$ and $N = 1000$ with 100 independent realizations. The solid black line shows analytic predictions of the Marchenko–Pastur law [Eq. (65)] which is exact in thermodynamic limit $N, M \to \infty$.

2.5. Spectral density via resolvent

Using the quadratic formula, we get

$$\bar{\nu} = \frac{z \pm \sqrt{z^2 - 4\sigma^2}}{2\sigma^2}. \tag{36}$$

To extract the spectrum, we make use of Eq. (3) to relate the spectral density to the imaginary part of the Green's function. After substituting $z = x - i0^+$, it is easy to convince oneself that the only way to get imaginary numbers here is to have $z = x$ with $|x| \leq 2\sigma$. This yields the expression

$$\rho_A(x) = \begin{cases} \frac{1}{2\pi\sigma^2}\sqrt{4\sigma^2 - x^2} & \text{if} \quad |x| \leq 2\sigma \\ 0 & \text{if} \quad |x| > 2\sigma \end{cases} \tag{37}$$

which is simply Wigner's semi-circle law.

3. Marchenko–Pastur distribution

We now derive the Marchenko–Pastur distribution. As before, we define a linear system of equations whose susceptibilities are related to the relevant Green's function. We will be interested in the spectral density of the ensemble of the $M \times M$ Wishart matrices $A = CC^T$ where C is an $M \times N$ matrix with entries $C_{i\alpha}$ ($i = 1, \ldots, M$ and $\alpha = 1, \ldots, N$) which are drawn from a normal distribution with mean and variance given by

$$\langle C_{i\alpha} \rangle = 0, \quad \left\langle C_{i\alpha} C_{j\beta} \right\rangle = \frac{\sigma^2}{N} \delta_{ij} \delta_{\alpha\beta} \tag{38}$$

Furthermore, we define the ratio

$$\gamma = \frac{M}{N}. \tag{39}$$

We will be interested in the limit $M, N \to \infty$ with γ fixed. A well known result in RMT is that the spectrum of such Wishart matrices is given by the Marchenko–Pastur distribution [28] (see Fig. 2). We re-derive this result with our method.

3.1. Resolvent equations

To apply our method, we start with the same system of equations used in the previous section to compute the Wigner semi-circle law, Eq. (18). Inserting the definition of the Wishart matrix $A = CC^T$, we get

$$z u_i = \sum_{j\alpha} C_{i\alpha} C_{j\alpha} u_j + a_i \tag{40}$$

where u_i are a set of N unknown variables, z is a constant, and a_i are a set of N of constant auxiliary variables (Note: indices represented by Roman letters are implied to range from 1 to N, while indices represented by Greek letters range from 1 to M).

Following the derivation in Section 2.1, it is clear that the susceptibility matrix

$$v_{ij}^{(u)} = \frac{\partial u_i}{\partial a_j} \tag{41}$$

is equivalent to the Green's function $G_A(z)$.

However, to apply our method, we require a system of equations that is linear in the elements of the constituent matrix C. Such a system of equations is found by rewriting Eq. (40) in terms of an extra set of M unknown variables v_α such that

$$\begin{aligned} zu_i &= \sum_\alpha C_{i\alpha} v_\alpha + a_i \\ v_\alpha &= \sum_j C_{j\alpha} u_j + b_\alpha \end{aligned} \tag{42}$$

where we have introduced an extra set of M constant auxiliary variables b_α to complement those in the first equation. We note that this system of equations is unique as the right hand side of the first equation must depend on the index i and thus the sum must range over the index α. We note that as long as a matrix can be written as a product of matrices such a system of equations can always be uniquely defined (see Section 4 for an example for a product of four matrices).

As before, we will be interested in small perturbations around the solutions to these equations. Therefore, we must consider an expanded set of susceptibilities,

$$\begin{aligned} v_{ij}^{(u)} &= \frac{\partial u_i}{\partial a_j}, \quad v_{\alpha j}^{(v)} = \frac{\partial v_\alpha}{\partial a_j}, \\ \chi_{i\beta}^{(u)} &= \frac{\partial u_i}{\partial b_\beta}, \quad \chi_{\alpha\beta}^{(v)} = \frac{\partial v_\alpha}{\partial b_\beta}, \end{aligned} \tag{43}$$

that measure the extent to which u_i and v_α change in response to a small perturbations in the constants a_i and b_α.

Just as $v_{ij}^{(u)}$ is the Green's function, the other susceptibility matrices can also be expressed in terms of z and C. Differentiating this system of equations with respect to the auxiliary variables a_j and b_α yields

$$\begin{aligned} zv_{ik}^{(u)} &= \sum_\alpha C_{i\alpha} v_{\alpha k}^{(n)} + \delta_{ik} \\ z\chi_{i\beta}^{(u)} &= \sum_\alpha C_{i\alpha} \chi_{\alpha\beta}^{(v)} \\ v_{\alpha k}^{(v)} &= \sum_j C_{j\alpha} v_{jk}^{(u)} \\ \chi_{\alpha\beta}^{(v)} &= \sum_j C_{j\alpha} \chi_{j\beta}^{(u)} + \delta_{\alpha\beta} \end{aligned} \tag{44}$$

which we can rewrite in matrix form as

$$\begin{pmatrix} zI_N & -C \\ -C^T & I_M \end{pmatrix} \begin{pmatrix} v^{(u)} & \chi^{(u)} \\ v^{(v)} & \chi^{(v)} \end{pmatrix} = \begin{pmatrix} I_N & 0 \\ 0 & I_M \end{pmatrix} \tag{45}$$

where I_N is the $N \times N$ identity matrix and I_M is the $M \times M$ identity matrix. Inverting this equation yields

$$\begin{pmatrix} v^{(u)} & \chi^{(u)} \\ v^{(v)} & \chi^{(v)} \end{pmatrix} = \begin{pmatrix} zI_N & -C \\ -C^T & I_M \end{pmatrix}^{-1} \tag{46}$$

We then use standard formulas for inverting block matrices [35] to get

$$\begin{aligned} v^{(u)} &= (zI_N - CC^T)^{-1} = \frac{1}{z - A} \\ \chi^{(u)} &= (zI_N - CC^T)^{-1} C \\ v^{(v)} &= C^T (zI_N - CC^T)^{-1} \\ \chi^{(v)} &= I_M + C^T (zI_N - CC^T)^{-1} C \end{aligned} \tag{47}$$

Comparing with Eq. (1) we see that $v^{(u)}$ is exactly the resolvent of the random matrix A as expected.

3.2. Cavity expansion

We once again calculate susceptibilities using the zero temperature cavity method. To do so, we introduce two new variables u_0 and v_0, and a new row and column to the matrix C denoted by $C_{0\alpha}$ and C_{i0}, respectively. In the presence of these new variables, the original $M + N$ equations are modified to

$$\begin{aligned} zu_i &= \sum_\alpha C_{i\alpha} v_\alpha + a_i + C_{i0} v_0 \\ v_\alpha &= \sum_j C_{j\alpha} u_j + b_\alpha + C_{0\alpha} u_0 \end{aligned} \tag{48}$$

Next, we interpret the terms $C_{i0}v_0$ and $C_{0\alpha}u_0$ as small perturbations to the constants a_i and b_α since the matrix elements of C scale as $1/N$,

$$\delta a_i = C_{i0}v_0, \quad \delta v_\alpha = C_{0\alpha}u_0. \tag{49}$$

We then perturbatively relate the solution to Eq. (48) in the presence of the new variables, u_0 and v_0, to the solutions of Eq. (42) without the new variables, denoted $u_{i\setminus 0}$ and $v_{\alpha\setminus 0}$, using the definitions of the susceptibilities [Eq. (43)],

$$\begin{aligned} u_i &\approx u_{i\setminus 0} + \sum_j \nu_{ij}^{(u)} C_{j0}v_0 + \sum_\beta \chi_{i\beta}^{(u)} C_{0\beta}u_0 \\ v_\alpha &\approx v_{\alpha\setminus 0} + \sum_j \nu_{\alpha j}^{(v)} C_{j0}v_0 + \sum_\beta \chi_{\alpha\beta}^{(v)} C_{0\beta}u_0. \end{aligned} \tag{50}$$

Now we consider the new equations for the two new variables,

$$\begin{aligned} zu_0 &= \sum_\alpha C_{0\alpha}v_\alpha + a_0 + C_{00}v_0 \\ v_0 &= \sum_j C_{j0}u_j + b_0 + C_{00}u_0. \end{aligned} \tag{51}$$

Substituting the expansions into Eq. (51) gives

$$\begin{aligned} zu_0 &= \sum_\alpha C_{0\alpha}v_{\alpha\setminus 0} + v_0 \sum_{\alpha j} \nu_{\alpha j}^{(v)} C_{0\alpha}C_{j0} + u_0 \sum_{\alpha\beta} \chi_{\alpha\beta}^{(v)} C_{0\alpha}C_{0\beta} + a_0 + C_{00}v_0 \\ v_0 &= \sum_j C_{j0}u_{j\setminus 0} + v_0 \sum_{jk} \nu_{jk}^{(u)} C_{j0}C_{k0} + u_0 \sum_{j\beta} \chi_{j\beta}^{(u)} C_{j0}C_{0\beta} + b_0 + C_{00}u_0. \end{aligned} \tag{52}$$

3.3. Approximation via central limit theorem

Similar to the approximation made in the derivation of the Wigner semi-circle law in Section 2.3, we approximate each of the large sums involving the susceptibilities matrices using the Central Limit Theorem. In particular, it is straightforward to show that the variance of each of these sums is small in the thermodynamic limit $N, M \to \infty$, allowing us to approximate each with just its mean with respect to the new row an column of C (those elements with at least one 0-valued index). First, we approximate the sums involving the square susceptibility matrices $\nu_{jk}^{(u)}$ and $\chi_{\alpha\beta}^{(v)}$ as

$$\begin{aligned} \sum_{jk} \nu_{jk}^{(u)} C_{j0}C_{k0} &\approx \sum_{jk} \nu_{jk}^{(u)} \left\langle C_{j0}C_{k0} \right\rangle = \sigma^2 \bar{\nu} \\ \sum_{\alpha\beta} \chi_{\alpha\beta}^{(v)} C_{0\alpha}C_{0\beta} &\approx \sum_{\alpha\beta} \chi_{\alpha\beta}^{(v)} \left\langle C_{0\alpha}C_{0\beta} \right\rangle = \sigma^2 \gamma \bar{\chi} \end{aligned} \tag{53}$$

where we have defined the traces to the susceptibilities as

$$\begin{aligned} \bar{\nu} &= \frac{1}{N} \sum_j \nu_{jj}^{(u)} \\ \bar{\chi} &= \frac{1}{M} \sum_\alpha \chi_{\alpha\alpha}^{(v)}. \end{aligned} \tag{54}$$

Each term in the other two sums contains a pair of independent elements of C, resulting in zero mean,

$$\begin{aligned} \sum_{\alpha j} \nu_{\alpha j}^{(v)} C_{0\alpha}C_{j0} &\approx \sum_{\alpha j} \nu_{\alpha j}^{(v)} \left\langle C_{0\alpha}C_{j0} \right\rangle = 0 \\ \sum_{j\beta} \chi_{j\beta}^{(u)} C_{j0}C_{0\beta} &\approx \sum_{j\beta} \chi_{j\beta}^{(u)} \left\langle C_{j0}C_{0\beta} \right\rangle = 0. \end{aligned} \tag{55}$$

As a result only the two square susceptibility matrices are required for this calculation.

3.4. Self-consistency equations for susceptibilities

Applying the approximations from the previous sections to Eq. (52) (and dropping the terms $C_{00}v_0$ and $C_{00}u_0$ since there scale as $1/N$) we get

$$\begin{aligned} zu_0 &= \sum_\alpha C_{0\alpha}v_{\alpha\setminus 0} + u_0\sigma^2\gamma\bar{\chi} + a_0 \\ v_0 &= \sum_j C_{j0}u_{j\setminus 0} + v_0\sigma^2\bar{\nu} + b_0 \end{aligned} \tag{56}$$

which can be rearranged to yield

$$\begin{aligned} u_0 &= \frac{\sum_\alpha C_{0\alpha}v_{\alpha\setminus 0} + a_0}{z - \sigma^2\gamma\bar{\chi}} \\ v_0 &= \frac{\sum_j C_{j0}u_{j\setminus 0} + b_0}{1 - \sigma^2\bar{\nu}}. \end{aligned} \tag{57}$$

From the cavity construction, we know that the traces of the susceptibility self-averaging, allowing us to write

$$\begin{aligned}\bar{\nu} &\approx \left\langle \nu_{00}^{(u)} \right\rangle = \left\langle \frac{\partial u_0}{\partial a_0} \right\rangle = \frac{1}{z - \sigma^2 \gamma \bar{\chi}} \\ \bar{\chi} &\approx \left\langle \chi_{00}^{(v)} \right\rangle = \left\langle \frac{\partial v_0}{\partial b_0} \right\rangle = \frac{1}{1 - \sigma^2 \bar{\nu}}.\end{aligned} \tag{58}$$

We combine these equations to get

$$\bar{\nu} = \frac{1}{z - \sigma^2 \frac{1}{1 - \gamma \sigma^2 \bar{\nu}}} \tag{59}$$

which gives us a quadratic equation for $\bar{\nu}$,

$$z\gamma\sigma^2 \bar{\nu}^2 - \left[z + \sigma^2(\gamma - 1)\right]\bar{\nu} + 1 = 0. \tag{60}$$

3.5. Spectral density via resolvent

Solving the quadratic equation, we obtain

$$\bar{\nu}(z) = \frac{z + \sigma^2(\gamma - 1)}{2z\gamma\sigma^2} \pm \frac{\sqrt{\left[z + \sigma^2(\gamma - 1)\right]^2 - 4z\gamma\sigma^2}}{2z\gamma\sigma^2}. \tag{61}$$

To extract the spectral density, we make use of Eq. (3). Notice that the imaginary part of the first term in the sum above for $z = x - i0^+$ is simply

$$\lim_{\epsilon \to 0^+} \frac{1}{2\pi}(1 - \gamma^{-1}) \frac{\epsilon}{x^2 + \epsilon^2}, \tag{62}$$

which we recognize as the definition of delta function at zero. For this reason, this term accounts for the spectral weight of the zero eigenvalues and hence is non-zero only if the Wishart matrix is not full rank (i.e., $\gamma > 1$).

Now consider the second term in Eq. (61). In order to have a non-zero imaginary part of $\bar{\nu}$ when $z = x - i0^+$, the discriminant must be negative. This allows us to calculate the upper and lower bounds for where $\rho(x)$ is non-zero by setting the discriminant in Eq. (61) to zero with $z = x$,

$$\left[x + \sigma^2(\gamma - 1)\right]^2 - 4x\gamma\sigma^2 = 0, \tag{63}$$

yielding,

$$\begin{aligned}x_{\min} &= (\gamma + 1)\sigma^2 - 2\sqrt{\gamma}\sigma^2 \\ x_{\max} &= (\gamma + 1)\sigma^2 + 2\sqrt{\gamma}\sigma^2.\end{aligned} \tag{64}$$

Combining this with Eq. (61), the contribution of this term to the spectral density is non-zero only on a finite region $[x_{\min}, x_{\max}]$. Together, these observations yield the following expression for the spectral density

$$\rho_A(x) = \begin{cases} (1 - \gamma^{-1})\delta(x) + \frac{1}{2\pi x\gamma\sigma^2}\sqrt{(x - x_{\min})(x_{\max} - x)}, & \text{if} \quad \gamma > 1 \\ \frac{1}{2\pi x\gamma\sigma^2}\sqrt{(x - x_{\min})(x_{\max} - x)}, & \text{if} \quad \gamma \leq 1. \end{cases} \tag{65}$$

This is exactly the Marchenko–Pastur distribution.

4. Wishart product matrices

Next, we would like to derive the spectrum for a special case of Wishart product matrices with our approach [36,37]. In the general case, these matrices take the form $A = CC^T$ where $C = B_1 B_2 \times \cdots \times B_n$ where each matrix B_i can be a rectangular matrix of a different size (such that the matrix multiplications are valid). In this section, we consider the spectral density of the ensemble of Wishart product matrices $A = CBB^T C^T$ where C and B are both $N \times N$ matrices whose entries, B_{ij} and C_{ij}, respectively, are independently drawn from normal distributions with mean and variances given by

$$\begin{aligned}\left\langle B_{ij} \right\rangle = 0, \quad \left\langle B_{ij} B_{kl} \right\rangle &= \frac{\sigma_B^2}{N} \delta_{ik} \delta_{jl} \\ \left\langle C_{ij} \right\rangle = 0, \quad \left\langle C_{ij} C_{kl} \right\rangle &= \frac{\sigma_C^2}{N} \delta_{ik} \delta_{jl} \\ \left\langle B_{ij} C_{kl} \right\rangle &= 0.\end{aligned} \tag{66}$$

4.1. Resolvent equations

Following Section 2.1, we begin by inserting the definition for A into Eq. (18),

$$zu_i = \sum_{jklm} C_{ij} B_{jk} B_{lk} C_{ml} u_m + a_i \tag{67}$$

where u_i $(i = 1, \ldots, N)$ are a set of unknown variables, z is a constant, and a_i are a set of constant auxiliary variables. Just as the previous section, it is clear that the Green's function $G_A(z)$ is given by the susceptibility matrix

$$\nu_{ij}^{(u)} = \frac{\partial u_i}{\partial a_j}. \tag{68}$$

Next, we follow the setup described for product matrices in Section 3.1, defining a new set of unknown variables, v_i, w_i, and x_i $(i = 1, \ldots, N)$, allowing us to decompose Eq. (67) such that each equation is linear in the elements of the random matrices C and B,

$$\begin{aligned} zu_i &= \sum_j C_{ij} v_j + a_i \\ v_i &= \sum_j B_{ij} w_j + b_i \\ w_i &= \sum_j B_{ji} x_j + c_i \\ x_i &= \sum_j C_{il} u_i + d_i. \end{aligned} \tag{69}$$

In each equation of the additional three equations, we have also introduced a new set of constant auxiliary variables, b_i, c_i, and d_i.

In order to explore small perturbations about the solutions to these equations, we define all possible susceptibility matrices with respect to the auxiliary variables,

$$\begin{aligned} \nu_{ij}^{(u)} &= \frac{\partial u_i}{\partial a_j}, \nu_{ij}^{(v)} = \frac{\partial v_i}{\partial a_j}, \nu_{ij}^{(w)} = \frac{\partial w_i}{\partial a_j}, \nu_{ij}^{(x)} = \frac{\partial x_i}{\partial a_j} \\ \chi_{ij}^{(u)} &= \frac{\partial u_i}{\partial b_j}, \chi_{ij}^{(v)} = \frac{\partial v_i}{\partial b_j}, \chi_{ij}^{(w)} = \frac{\partial w_i}{\partial b_j}, \chi_{ij}^{(x)} = \frac{\partial x_i}{\partial b_j} \\ \phi_{ij}^{(u)} &= \frac{\partial u_i}{\partial c_j}, \phi_{ij}^{(v)} = \frac{\partial v_i}{\partial c_j}, \phi_{ij}^{(w)} = \frac{\partial w_i}{\partial c_j}, \phi_{ij}^{(x)} = \frac{\partial x_i}{\partial c_j} \\ \omega_{ij}^{(u)} &= \frac{\partial u_i}{\partial d_j}, \omega_{ij}^{(v)} = \frac{\partial v_i}{\partial d_j}, \omega_{ij}^{(w)} = \frac{\partial w_i}{\partial d_j}, \omega_{ij}^{(x)} = \frac{\partial x_i}{\partial d_j}. \end{aligned} \tag{70}$$

As in the previous section, only a small subset of these susceptibilities will be relevant.

4.2. Cavity expansion

Next, we begin the zero-cavity method by introducing a four variables, u_0, v_0, w_0, and x_0, and a new row and column to each matrices denoted by C_{i0}, C_{0j}, B_{i0}, and B_{0j}. In the presence of these new variables, the original $4N$ equations [Eq. (69)] become

$$\begin{aligned} zu_i &= \sum_j C_{ij} v_j + a_i + C_{i0} v_0 \\ v_i &= \sum_j B_{ij} w_j + b_i + B_{i0} w_0 \\ w_i &= \sum_j B_{ji} x_j + c_i + B_{0i} x_0 \\ x_i &= \sum_j C_{ji} u_j + d_i + C_{0i} u_0. \end{aligned} \tag{71}$$

From here, we interpret the extra terms as small (order $1/N$) perturbations to the auxiliary variables,

$$\begin{aligned} \delta a_i &= C_{i0} v_0, \quad \delta b_i = B_{i0} w_0 \\ \delta c_i &= B_{0i} x_0, \quad \delta d_i = C_{0i} u_0, \end{aligned} \tag{72}$$

allowing us to perturbatively relate the solutions of these equations with the new variables to those of Eq. (69), denoted $u_{i\backslash 0}$, $v_{i\backslash 0}$, $w_{i\backslash 0}$, $x_{i\backslash 0}$. This gives us

$$\begin{aligned} u_i &\approx u_{i\backslash 0} + \sum_j \nu_{ij}^{(u)} C_{i0} v_0 + \sum_j \chi_{ij}^{(u)} B_{j0} w_0 + \sum_j \phi_{ij}^{(u)} B_{0j} x_0 + \sum_j \omega_{ij}^{(u)} C_{0j} u_0 \\ v_i &\approx v_{i\backslash 0} + \sum_j \nu_{ij}^{(v)} C_{i0} v_0 + \sum_j \chi_{ij}^{(v)} B_{j0} w_0 + \sum_j \phi_{ij}^{(v)} B_{0j} x_0 + \sum_j \omega_{ij}^{(v)} C_{0j} u_0 \\ w_i &\approx w_{i\backslash 0} + \sum_j \nu_{ij}^{(w)} C_{i0} v_0 + \sum_j \chi_{ij}^{(w)} B_{j0} w_0 + \sum_j \phi_{ij}^{(w)} B_{0j} x_0 + \sum_j \omega_{ij}^{(w)} C_{0j} u_0 \\ x_i &\approx x_{i\backslash 0} + \sum_j \nu_{ij}^{(x)} C_{i0} v_0 + \sum_j \chi_{ij}^{(x)} B_{j0} w_0 + \sum_j \phi_{ij}^{(x)} B_{0j} x_0 + \sum_j \omega_{ij}^{(x)} C_{0j} u_0. \end{aligned} \tag{73}$$

Next, we will consider the additional equations for the new variables, given by

$$\begin{aligned} zu_0 &= \sum_j C_{0j} v_j + a_0 + C_{00} v_0 \\ v_0 &= \sum_j B_{0j} w_j + b_0 + B_{00} w_0 \\ w_0 &= \sum_j B_{j0} x_j + c_0 + B_{00} x_0 \\ x_0 &= \sum_j C_{j0} u_j + d_0 + C_{00} u_0. \end{aligned} \tag{74}$$

4.3. Approximation via central limit theorem

Following the procedure described in the previous sections, we substitute the expansions in Eq. (73) into the equations for the new variables in Eq. (74). The resulting set of equations contains many sums over large numbers of random variables. In particular, for each sum containing one of the susceptibility matrices, we make use of the fact that the susceptibilities are statistically independent of the new rows and columns of C and B. Using the central limit theorem, each sum can be shown to be dominated by its mean. Four of these sums can be shown to be nonzero, following the same form as those in Eq. (53),

$$\sum_{jk} v^{(u)}_{jk} C_{j0} C_{k0} \approx \sigma_C^2 \bar{v} \tag{75}$$

$$\sum_{jk} \omega^{(v)} C_{0j} C_{0k} \approx \sigma_C^2 \bar{\omega} \tag{76}$$

$$\sum_{jk} \phi^{(w)} B_{0j} B_{0k} \approx \sigma_B^2 \bar{\phi} \tag{77}$$

$$\sum_{jk} \chi^{(x)}_{ij} B_{j0} B_{k0} \approx \sigma_B^2 \bar{\chi} \tag{78}$$

where we have defined the traces of the susceptibilities matrices as

$$\begin{aligned} \bar{v} &= \frac{1}{N} \sum_j v^{(u)}_{jj}, \quad \bar{\chi} = \frac{1}{N} \sum_j \chi^{(x)}_{jj} \\ \bar{\phi} &= \frac{1}{N} \sum_j \phi^{(w)}_{jj}, \quad \bar{\omega} = \frac{1}{N} \sum_j \omega^{(v)}_{jj}. \end{aligned} \tag{79}$$

It can easily be shown that the remaining twelve sums containing one of the susceptibilities are all similar in form to those in Eq. (55) and are therefore approximately zero in the thermodynamic limit $N \to \infty$.

4.4. Self-consistency equations for susceptibilities

Applying the approximations from the previous section and neglecting the additional terms of order $1/N$, Eq. (74) becomes

$$\begin{aligned} zu_0 &\approx \sum_j C_{0j} v_{j\setminus 0} + u_0 \sigma_C^2 \bar{\omega} + a_0 \\ v_0 &\approx \sum_j B_{0j} w_{j\setminus 0} + x_0 \sigma_B^2 \bar{\phi} + b_0 \\ w_0 &\approx \sum_j B_{j0} x_{j\setminus 0} + w_0 \sigma_B^2 \bar{\chi} + c_0 \\ x_0 &\approx \sum_j C_{j0} u_{j\setminus 0} + v_0 \sigma_C^2 \bar{v} + d_0. \end{aligned} \tag{80}$$

Solving these equations for the new variables, we find

$$\begin{aligned} u_0 &= \frac{\sum_j C_{0j} v_{j\setminus 0} + a_0}{z - \sigma_C^2 \bar{\omega}} \\ v_0 &= \frac{\sum_j B_{0j} w_{j\setminus 0} + b_0 + \sigma_B^2 \bar{\phi} \left(\sum_j C_{j0} u_{j\setminus 0} + d_0 \right)}{1 - \sigma_B^2 \sigma_C^2 \bar{v} \bar{\phi}} \\ w_0 &= \frac{\sum_j B_{j0} x_{j\setminus 0} + c_0}{1 - \sigma_B^2 \bar{\chi}} \\ x_0 &= \frac{\sigma_C^2 \bar{v} \left(\sum_j B_{0j} w_{j\setminus 0} + b_0 \right) + \sum_j C_{j0} u_{j\setminus 0} + d_0}{1 - \sigma_B^2 \sigma_C^2 \bar{v} \bar{\phi}}. \end{aligned} \tag{81}$$

Next, we approximate the traces of the susceptibility matrices as the average over a single element, giving us a set of self-consistent equations

$$\begin{aligned}
\bar{\nu} &\approx \left\langle \nu_{00}^{(u)} \right\rangle = \left\langle \frac{\partial u_0}{\partial a_0} \right\rangle = \frac{1}{z - \sigma_C^2 \bar{\omega}} \\
\bar{\chi} &\approx \left\langle \chi_{00}^{(x)} \right\rangle = \left\langle \frac{\partial x_0}{\partial b_0} \right\rangle = \frac{\sigma_C^2 \bar{\nu}}{1 - \sigma_B^2 \sigma_C^2 \bar{\nu} \bar{\phi}} \\
\bar{\phi} &\approx \left\langle \phi_{00}^{(w)} \right\rangle = \left\langle \frac{\partial w_0}{\partial c_0} \right\rangle = \frac{1}{1 - \sigma_B^2 \bar{\chi}} \\
\bar{\omega} &\approx \left\langle \bar{\omega}_{00}^{(v)} \right\rangle = \left\langle \frac{\partial v_0}{\partial d_0} \right\rangle = \frac{\sigma_B^2 \bar{\phi}}{1 - \sigma_B^2 \sigma_C^2 \bar{\nu} \bar{\phi}}.
\end{aligned} \tag{82}$$

Combining these equations, we arrive at a cubic equation for $\bar{\nu}$ of the form

$$\sigma_B^2 \sigma_C^2 z^2 \bar{\nu}^3 - z\bar{\nu} + 1 = 0. \tag{83}$$

4.5. Spectral density via resolvent

To solve this cubic equation, we first rewrite it in terms of $z\bar{\nu}$,

$$0 = (z\bar{\nu})^3 - \frac{z}{\sigma_B^2 \sigma_C^2}(z\bar{\nu}) + \frac{z}{\sigma_B^2 \sigma_C^2}, \tag{84}$$

allowing to express the three solutions in general form as

$$\begin{aligned}
z\bar{\nu}^{(1)} &= S + T \\
z\bar{\nu}^{(2)} &= -\frac{1}{2}(S+T) + \frac{1}{2} i\sqrt{3}(S-T) \\
z\bar{\nu}^{(3)} &= -\frac{1}{2}(S+T) - \frac{1}{2} i\sqrt{3}(S-T)
\end{aligned} \tag{85}$$

where

$$\begin{aligned}
S &= \sqrt[3]{R + \sqrt{D}} \\
T &= \sqrt[3]{R - \sqrt{D}} \\
D &= R^2 - Q^3
\end{aligned} \tag{86}$$

with

$$\begin{aligned}
Q &= \frac{1}{3} \frac{z}{\sigma_B^2 \sigma_C^2} \\
R &= -\frac{1}{2} \frac{z}{\sigma_B^2 \sigma_C^2}.
\end{aligned} \tag{87}$$

In writing these expressions, we have made us of standard mathematical identities for the roots of a cubic equation.

We then make use of Eq. (3) to extract the spectral density. After substituting $z = x - i0^+$, it is possible to convince oneself that the spectral density is given by the imaginary part of solution $\bar{\nu}^{(2)}$ when the polynomial discriminant D is positive. To solve for the bounds of the distribution, we solve for the roots of D after setting $z = x$,

$$D = \frac{1}{4}\left(\frac{x}{\sigma_B^2 \sigma_C^2}\right)^2 - \frac{1}{27}\left(\frac{x}{\sigma_B^2 \sigma_C^2}\right)^3 = 0. \tag{88}$$

The limiting eigenvalues are then

$$\begin{aligned}
x_{\min} &= 0 \\
x_{\max} &= \frac{27}{4} \sigma_B^2 \sigma_C^2.
\end{aligned} \tag{89}$$

Finally, the spectral density is given by

$$\rho(x) = \frac{\sqrt{3}}{2\pi x}\left[S_+\left(\frac{x}{\sigma_B^2 \sigma_C^2}\right) - S_-\left(\frac{x}{\sigma_B^2 \sigma_C^2}\right)\right] \tag{90}$$

where

$$S_\pm(a) = \sqrt[3]{\frac{1}{2}a \pm \sqrt{\frac{1}{27}a^2\left(\frac{27}{4} - a\right)}}. \tag{91}$$

Numerical checks of these expression is show in Fig. 3. The same result has been reported in [37].

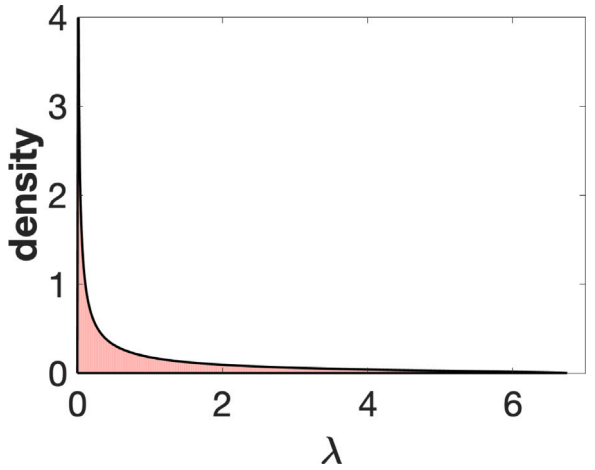

Fig. 3. The spectrum of a Wishart product matrix of the form $A = (CB)(CB)^T$ where C and B are $N \times N$ matrices with entries *i.i.d* with mean zero and variances σ_C^2/N and σ_B^2/N, respectively, where $\sigma_C^2 = \sigma_B^2 = 1$ and $N = 1000$ with 100 independent realizations. The solid black line shows the analytic prediction of Eq. (90) which is exact in thermodynamic limit $N \to \infty$.

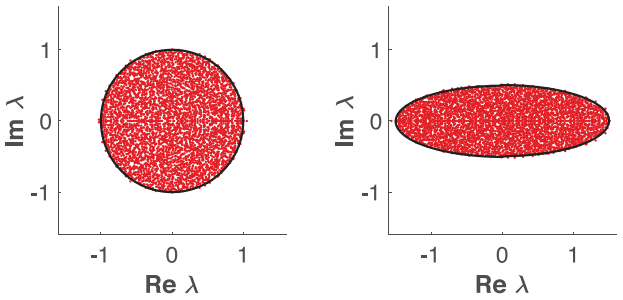

Fig. 4. Circle and elliptic law for real matrices. The eigenvalues of a real $N \times N$ matrix A with entries A_{ij} with mean zero and $\langle A_{ij} A_{kl} \rangle = \frac{\sigma^2}{N}\delta_{ik}\delta_{jl} + \frac{\zeta\sigma^2}{N}\delta_{il}\delta_{jk}$. (Left) Spectrum for a matrix with $N = 1000$, $\sigma = 1$, and $\gamma = 0$. The solid black line shows analytic predictions of the circle law which is exact in thermodynamic limit $N \to \infty$. (Right) The spectrum for a matrix with $N = 1000$, $\sigma = 1$, and $\gamma = 0.5$. The solid black line shows analytic predictions of the elliptic law which is exact in thermodynamic limit $N \to \infty$. There are 5 independent realizations in both figures.

5. The circle and elliptic laws

In the previous sections, we considered symmetric matrices which have real eigenvalues. We now show how to apply our approach in calculating the spectral density of non-symmetric real matrices where the eigenvalues have both real and imaginary parts. We consider the ensemble of real $N \times N$ matrices A with elements A_{ij} such that

$$\langle A_{ij} \rangle = 0, \quad \langle A_{ij} A_{kl} \rangle = \frac{\sigma^2}{N}\delta_{ij}\delta_{kl} + \frac{\zeta\sigma^2}{N}\delta_{il}\delta_{jk}. \tag{92}$$

The spectral densities of such ensembles are known to be described by the Girko circle law when $\zeta = 0$ [29] and more generally, the elliptic law when $\zeta \neq 0$ (see Fig. 4).

5.1. Resolvent equations

We now derive both these spectral densities. Inspired by Eq. (12), we consider the $2N \times 2N$ dimensional matrix

$$H(z) = \begin{pmatrix} 0 & A - zI \\ -A^* + z^* I & 0 \end{pmatrix} \tag{93}$$

where z is a constant and I is the $N \times N$ identity matrix. In analogy to Eq. (18), we construct a system of $2N$ real equations for the variables x_i and y_i $(i = 1, \ldots, N)$ of the form

$$\eta \begin{pmatrix} \vec{x} \\ \vec{y} \end{pmatrix} = H(z) \begin{pmatrix} \vec{x} \\ \vec{y} \end{pmatrix} + \begin{pmatrix} \vec{a} \\ \vec{b} \end{pmatrix} \tag{94}$$

where η is a constant and a_i and b_i are $2N$ real-valued auxiliary variables. We rewrite these equations in component form as

$$\begin{aligned}\eta x_i &= \sum_j A_{ij} y_j - z y_i + a_i \\ \eta y_i &= -\sum_j A^*_{ij} x_j + z^* x_i + b_i.\end{aligned} \tag{95}$$

As before, we define a set of susceptibilities,

$$\begin{aligned}\chi^{(x)}_{ij} &= \frac{\partial x_i}{\partial a_j}, \quad \nu^{(y)}_{ij} = \frac{\partial y_i}{\partial a_j}, \\ \nu^{(x)}_{ij} &= \frac{\partial x_i}{\partial b_j}, \quad \chi^{(y)}_{ij} = \frac{\partial y_i}{\partial b_j},\end{aligned} \tag{96}$$

that measure how x_i and y_j change in response to a small changes in the constants a_i and b_j. Taking the derivative of Eq. (95) with respect to b_k and a_k gives

$$\begin{aligned}\eta \chi^{(x)}_{ik} &= \sum_j A_{ij} \nu^{(y)}_{jk} - z \nu^{(y)}_{ik} + \delta_{ik} \\ \eta \nu^{(x)}_{ik} &= \sum_j A_{ij} \chi^{(y)}_{jk} - z \chi^{(y)}_{ik} \\ \eta \nu^{(y)}_{ik} &= -\sum_j A^*_{ij} \chi^{(x)}_{jk} + z^* \chi^{(x)}_{ik} \\ \eta \chi^{(y)}_{ik} &= -\sum_j A^*_{ij} \nu^{(x)}_{jk} + z^* \nu^{(x)}_{ik} + \delta_{ik}.\end{aligned} \tag{97}$$

We then rewrite this in matrix form as

$$\begin{pmatrix} \eta I & zI - A \\ -z^* I + A^* & \eta I \end{pmatrix} \begin{pmatrix} \chi^{(x)} & \nu^{(x)} \\ \nu^{(y)} & \chi^{(y)} \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \tag{98}$$

Notice that the equations for the susceptibilities do not depend on the value of a_i and b_i. This implies that they are independent of the exact values of a_i and b_i and should be valid even for special case where all the constants are chosen to be identical, $a_i = b_i = c$ for all i. In this case, there is a hidden symmetry in our system of linear equations, Eq. (95), implying that the four susceptibilities are not independent, but instead related by complex conjugation. In light of these observations, it is easy to convince oneself that we must have $\chi^{(x)} = \chi^{(y)} = \chi$, $\nu^{(x)} = \nu$ and $\nu^{(y)} = -\nu^*$ where ν^* denotes the Hermitian conjugate of ν. This implies that we can rewrite our equations in the simpler form

$$\begin{pmatrix} \eta I & zI - A \\ -z^* I + A^* & \eta I \end{pmatrix} \begin{pmatrix} \chi & \nu \\ -\nu^* & \chi \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}. \tag{99}$$

Inverting this equation we see that the susceptibility matrices correspond exactly to the complex Green's function of Eq. (13),

$$\mathcal{G}_A(\eta) = \begin{pmatrix} \chi & \nu \\ -\nu^* & \chi \end{pmatrix}. \tag{100}$$

We can use these expressions to calculate the spectral function of A using Eq. (15).

5.2. Susceptibilities via cavity method

Once again we calculate the susceptibilities using the cavity method. To do so, we introduce two additional variables x_0 and y_0 and an additional row A_{0j} and column A_{i0} to the matrix A. With the addition of these new variables, Eq. (95) for x_i and y_i is modified to

$$\begin{aligned}\eta x_i &= \sum_j A_{ij} y_j - z y_i + a_i + A_{i0} y_0 \\ \eta y_i &= -\sum_j A^*_{ij} x_j + z^* x_i + b_i - A^*_{i0} x_0.\end{aligned} \tag{101}$$

As before, we treat the effect of these two extra variables as small perturbations since the additional terms in the equation above scale as $1/N$. In particular, we relate the solutions to Eq. (101) (denoted by x_i and y_i) to the solutions to Eq. (101) without the new variables x_0 and y_0 (denoted by $x_{i\backslash 0}$ and $y_{i\backslash 0}$) using the definition of the susceptibilities [Eq. (96)],

$$\begin{aligned}x_j &= x_{j\backslash 0} - \sum_k \nu^{(x)}_{jk} A^*_{k0} x_0 + \sum_k \chi^{(x)}_{jk} y A_{k0} y_0 \\ y_j &= y_{j\backslash 0} - \sum_k \chi^{(y)}_{jk} A^*_{k0} x_0 + \sum_k \nu^{(y)}_{jk} A_{k0} y_0,\end{aligned} \tag{102}$$

where in writing this we have made use of the relationships between the various susceptibilities noted above.

In the cavity method, there are also two additional equations for the new variables x_0 and y_0 given by

$$\begin{aligned}\eta x_0 &= \sum_j A_{0j} y_j - z y_0 + a_0 + A_{00} y_0 \\ \eta y_0 &= -\sum_j A^*_{0j} x_j + z^* x_0 + b_0 - A^*_{00} x_0.\end{aligned} \tag{103}$$

Substituting in the expansions above gives

$$\begin{aligned}\eta x_0 &= \sum_j A_{0j} y_{j\setminus 0} - \sum_{jk} \chi^{(y)}_{jk} A_{0j} A^*_{k0} x_0 + \sum_{jk} \nu^{(y)}_{jk} A_{0j} A_{k0} y_0 - z y_0 + a_0 + A_{00} y_0 \\ \eta y_0 &= -\sum_j A^*_{0j} x_{j\setminus 0} + \sum_{jk} \nu^{(x)}_{jk} A^*_{0j} A^*_{k0} x_0 - \sum_{jk} \chi^{(x)}_{jk} A^*_{0j} A_{k0} y_0 + z^* x_0 + b_0 - A^*_{00} x_0.\end{aligned} \tag{104}$$

Next, we approximate each sum containing a susceptibility matrix with its mean (with respect to the new row and column of A) using the central limit theorem in the large N limit (see previous sections). Using Eq. (92) and noting that $A^*_{ij} = A_{ji}$ for real matrices and relationships between susceptibilities, we get

$$\begin{aligned}\eta x_0 &= \sum_j A_{0j} y_{j\setminus 0} - x_0 \sigma^2 \bar{\chi}^{(y)} + y_0 \zeta \sigma^2 \bar{\nu}^{(y)} - z y_0 + a_0 \\ \eta y_0 &= -\sum_j A_{j0} x_{j\setminus 0} + x_0 \zeta \sigma^2 \bar{\nu}^{(x)} - y_0 \sigma^2 \bar{\chi}^{(x)} + z^* x_0 + b_0,\end{aligned} \tag{105}$$

where the bar on the susceptibility denotes the trace. We are now in a position to solve for $\bar{\nu}^{(y)}$ and calculate the spectral density.

5.3. Circle law

Let us first focus on the special case where the entries are completely decorrelated with $\zeta = 0$. In this case, we know that the spectrum is described by the Girko circle law. To derive this, we use Eq. (105) with $\zeta = 0$ (noting that $\bar{\chi}^{(x)} = \bar{\chi}^{(y)} = \bar{\chi}$),

$$\begin{aligned}\eta x_0 &= \sum_j A_{0j} y_{j\setminus 0} - x_0 \sigma^2 \bar{\chi} - z y_0 + a_0 \\ \eta y_0 &= -\sum_j A_{j0} x_{j\setminus 0} - y_0 \sigma^2 \bar{\chi} + z^* x_0 + b_0.\end{aligned} \tag{106}$$

We solve this equation for x_0 to get

$$y_0 = \frac{(\eta + \sigma^2 \bar{\chi})\left(\sum_j A_{j0} x_{j\setminus 0} + b_0\right) + z^*\left(\sum_j A_{0j} y_{j\setminus 0} + a_0\right)}{|z|^2 + (\eta + \sigma^2 \bar{\chi})^2} \tag{107}$$

By definition of the susceptibilities and self-consistency of the mean field cavity equations, we know that

$$\bar{\chi} = \left\langle \frac{\partial y_0}{\partial b_0} \right\rangle = \frac{(\eta + \sigma^2 \bar{\chi})}{|z|^2 + (\eta + \sigma^2 \bar{\chi})^2}, \tag{108}$$

and

$$\bar{\nu}^* = -\bar{\nu}^{(y)} = -\left\langle \frac{\partial y_0}{\partial a_0} \right\rangle = -\frac{z^*}{|z|^2 + (\eta + \sigma^2 \bar{\chi})^2}. \tag{109}$$

Next, we examine the special case where $\eta = 0$. We further simplify this expression by exploiting the fact that Eq. (108) gives a cubic self-consistency equation for χ. When $\eta = 0$, we get

$$\sigma^4 \bar{\chi}^3 + \bar{\chi}(|z|^2 - \sigma^2) = 0. \tag{110}$$

which has two solutions:

$$\bar{\chi} = 0 \quad \text{or} \quad \sigma^4 \bar{\chi}^2 + |z|^2 = \sigma^2. \tag{111}$$

Substituting the first solution $\bar{\chi} = 0$ into the formula for the spectral density gives

$$\begin{aligned}\rho_A(u, v) &= -\frac{1}{\pi} \partial_{z^*} \bar{\nu}^*|_{\eta=0} = \frac{1}{\pi} \partial_{z^*} \frac{z^*}{|z|^2 + \sigma^4 \bar{\chi}^2} \\ &= \frac{1}{\pi} \partial_{z^*} \frac{z^*}{|z|^2} = \frac{1}{\pi} \partial_{z^*} \frac{1}{z} = 0\end{aligned} \tag{112}$$

while the second solution yields

$$\rho_A(u, v) = \frac{1}{\pi} \partial_{z^*} \frac{z^*}{|z|^2 + \sigma^4 \bar{\chi}^2} = \frac{1}{\pi} \partial_{z^*} \frac{z^*}{\sigma^2} = \frac{1}{\pi \sigma^2}. \tag{113}$$

All together, we find

$$\rho_A(u, v) = \begin{cases} \frac{1}{\pi \sigma^2} & \sigma^4 \bar{\chi}^2 + |z|^2 = \sigma^2 \\ 0 & \text{if} \quad \bar{\chi} = 0. \end{cases} \tag{114}$$

We would like translate this into a condition on z not $\bar{\chi}$. To do so, we note that $\sigma^4\bar{\chi}^2 + |z|^2 = \sigma^2$ implies that $|z|^2 \leq |\sigma|^2$. Since the two solutions must match at $\chi = 0$, this implies that the density takes the form

$$\rho_A(u,v) = \begin{cases} \frac{1}{\pi\sigma^2} & \text{if} \quad |z|^2 \leq \sigma^2 \\ 0 & \text{if} \quad |z|^2 \geq \sigma^2, \end{cases} \tag{115}$$

which is precisely the circle law.

5.4. Elliptic law

We now generalize this basic calculation to derive the Ellipse law for correlated matrices where $\zeta \neq 0$. We start once again with Eq. (105) and solve for y_0 to get

$$y_0 = \frac{(z^* + \zeta\sigma^2\bar{v})\Big(\sum_j A_{0j}y_{j\setminus 0} + a_0\Big) + \big(\eta + \sigma^2\bar{\chi}\big)\Big(\sum_j A_{j0}x_{j\setminus 0} + b_0\Big)}{|(z + \zeta\sigma^2\bar{v}^*)|^2 + (\eta + \sigma^2\bar{\chi})^2}. \tag{116}$$

We then use the usual cavity arguments to write

$$\bar{\chi} = \left\langle \frac{\partial y_0}{\partial b_0} \right\rangle = \frac{(\eta + \sigma^2\bar{\chi})}{|(z + \zeta\sigma^2\bar{v}^*)|^2 + (\eta + \sigma^2\bar{\chi})^2} \tag{117}$$

$$\bar{v}^{(y)} = \left\langle \frac{\partial y_0}{\partial a_0} \right\rangle = \frac{z^* + \zeta\sigma^2\bar{v}}{|(z + \zeta\sigma^2\bar{v}^*)|^2 + (\eta + \sigma^2\bar{\chi})^2} \tag{118}$$

To calculate the density, once again we will make use of Eq. (15). For $\eta = 0$, notice the equation for $\bar{\chi}$ has two solutions

$$\bar{\chi} = 0 \quad \text{or} \quad |(z + \zeta\sigma^2\bar{v}^*)|^2 + \sigma^4\bar{\chi}^2 = \sigma^2. \tag{119}$$

Plugging this into the expressions above yields (using the identity $\bar{v}^* = -\bar{v}^{(y)}$)

$$\bar{v}^* = \begin{cases} -\frac{1}{(z+\zeta\sigma^2\bar{v}^*)} & \text{if} \quad \chi = 0 \\ -\frac{z^*+\zeta\sigma^2\bar{v}}{\sigma^2} & \text{if} \quad |z + \zeta\sigma^2\bar{v}^*|^2 + \sigma^4\bar{\chi}^2 = \sigma^2, \end{cases} \tag{120}$$

When $\chi = 0$, the top expression in the equation above implies that $\bar{v}^*$ is an analytic function of z and does not depend on z^*. Hence, from Eq. (15) we conclude that when $\chi = 0$ that $\rho_A(u,v) = 0$. When $|z + \zeta\sigma^2\bar{v}^*|^2 + \sigma^4\bar{\chi}^2 = \sigma^2$, we can easily solve for $\bar{v}^*$ by combining the bottom equation with its complex conjugate to get

$$\bar{v}^* = -\frac{z^* - \zeta z}{\sigma^2(1-\zeta^2)} \tag{121}$$

Using Eq. (15), we conclude that for this solution the density is then given by $\rho_A(x,y) = \frac{1}{\pi\sigma^2(1-\zeta^2)}$. Summarizing, we find that

$$\rho_A(u,v) = \begin{cases} \frac{1}{\pi\sigma^2(1-\zeta^2)} & \text{if} \quad |z + \zeta\sigma^2\bar{v}^*|^2 + \sigma^4\bar{\chi}^2 = \sigma^2 \\ 0 & \text{otherwise} \end{cases} \tag{122}$$

Notice that $|z + \zeta\sigma^2\bar{v}^*|^2 + \sigma^4\bar{\chi}^2 = \sigma^2$ implies that $|z + \zeta\sigma^2\bar{v}^*|^2 \leq \sigma^2$. Plugging in the explicit expression for $\bar{v}^*$ into this inequality and writing $z = u + iv$ allows us to rewrite this inequality in terms of u and v as

$$\frac{u^2}{(1+\zeta)^2} + \frac{v^2}{(1-\zeta)^2} \leq \sigma^2 \tag{123}$$

This is the precisely the elliptic law first derived by Ginbre and Girko [30]. To summarize, for a general ζ, the spectral density is given by

$$\rho_A(u,v) = \begin{cases} \frac{1}{\pi\sigma^2(1-\zeta^2)} & \text{if} \quad \frac{u^2}{(1+\zeta)^2} + \frac{v^2}{(1-\zeta)^2} \leq \sigma^2 \\ 0 & \text{otherwise} \end{cases} \tag{124}$$

When $\zeta = 0$, these reduces to the circle law derived above.

6. Conclusion

In this paper we present a simple, mean-field approach for calculating spectral densities for a variety of random matrix ensembles in the thermodynamic limit. The key idea of our approach is to construct a system of random linear equations and ask how the solution to these equations change in response to small perturbations. We show that susceptibility corresponding to these perturbations are directly related to Green's functions used to calculate spectral densities.

We have illustrated the generality and power of this method by providing simple derivations of the Wigner Semi-circle Law for symmetric matrices, the Marchenko–Pastur Law for Wishart matrices, the spectral density of a simple product Wishart matrix, and the Circle and elliptic laws for real random matrices. Our approach is useful for directly relating the sensitivity of linear equilibrium systems to results in RMT. For example, we have found our method can offer a new perspective for understanding

problems in ecology [13,15,16], machine learning [38], and statistical inference problems such as compressed sensing [39], where zero-temperature cavity calculations have been used to identify phase transitions. Furthermore, while not shown here, this method can also be applied to various types of nonlinear matrices, which has been explored in other contexts [15,38].

CRediT authorship contribution statement

Wenping Cui: Conceptualization, Formal analysis, Software, Validation, Visualization, Writing – original draft, Writing – review & editing. **Jason W. Rocks:** Conceptualization, Formal analysis, Software, Validation, Visualization, Writing – original draft, Writing – review & editing. **Pankaj Mehta:** Conceptualization, Formal analysis, Funding acquisition, Investigation, Project administration, Resources, Supervision, Validation, Visualization, Writing – original draft, Writing – review & editing.

Declaration of competing interest

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests: Pankaj Mehta reports financial support was provided by National Institutes of Health. Pankaj Mehta reports financial support was provided by Simons Foundation.

Data availability

No data was used for the research described in the article.

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